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A Construction of Lower-Potent Quasi-Antiorders in Semigroup with Apartness

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Abstract

In the present paper, we describe a construction of lower-potent positive quasi-antiorder in semigroup with apartness.

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1 Preliminaries and Introduction

This investigation, in Bishop's constructive mathematics in sense of well-known books [1, 2], [5] and Romano's papers [7 - 10], is continuation of forthcoming Crvenkovic and Romano's paper [3], and Romano's papers [11, 12]. Bishop's constructive mathematics is develop on Constructive Logic (or Intuitionistic Logic ([15])) - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in Constructive Logic the 'Double negation Law',

$$P \Longleftrightarrow \neg\neg P$$

does not hold, but the following implication

$$P \Longrightarrow \neg\neg P$$

holds even in Minimal Logic.

Let $(S, =, \neq)$ be a set. The relation $\neq$ is a binary relation on S, which satisfies the following properties:

$$\neg(x \neq x),$$

$$x \neq y \Longrightarrow y \neq x,$$

$$x \neq z \Longrightarrow x \neq y \vee y \neq z,$$

$$x \neq y \wedge y = z \Longrightarrow x \neq z.$$

Following Heyting, it called *apartness*. Let Y be a subset of S and $x \in S$. Following Bridges, by $x \bowtie Y$ we denote $(\forall y \in Y)(y \neq x)$ and by Y^C we denote subset $\{x \in S : x \bowtie Y\}$ - the strong compliment of Y in S. The subset Y of S is a *strongly extensional* ([13]) in S if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$.

A relation q on S is a *coequality* relation on S if and only if it is consistent, symmetric and cotransitive ([9]). Let $(S, =, \neq, \cdot)$ be a semigroup with apartness. A relation τ on S is a *quasi-antiorder* ([10.11]) on S if

$$\tau \subseteq \neq,$$

$$(\forall x, y, z \in S)((x, z) \in \tau \Longrightarrow (x, y) \in \tau \vee (y, z) \in \tau)$$

It τ is a quasi-antiorder on S compatible with the semigroup operation in S, then ([11]) the relation $q = \tau \cup \tau^{-1}$ is an anticongruence on S. Firstly, the relation $q^C = \{(x, y) \in S \times S : (x, y) \bowtie q\}$ is a congruence on S compatible with q, in the following sense $q \circ q^C \subseteq q$ and $q^C \circ q \subseteq q$ ([9], Theorem 1). We can construct the semigroup $S/(q^C, q) = \{aq^C : a \in S\}$ ([9], Theorem 2) with

$aq^C = bq^C \Longleftrightarrow (a, b) \bowtie q$, $aq^C \neq bq^C \Longleftrightarrow (a, b) \in q$, $aq^C \cdot bq^C = (ab)q^C$.

We can also construct the semigroup $S/q = \{aq : a \in S\}$ ([9],Theorem 3) with

$$aq = b \Longleftrightarrow (a, b) \bowtie q,\ aq \neq b \Longleftrightarrow (a, b) \in q,\ aq \cdot bq = (ab)q$$

It is easily to establish the isomorphism $S/(q^C, q) \cong S/q$.

A subset A of a semigroup S is *consistent* if for a, b of S $ab \in A$ implies $a \in A$ and $b \in A$. it is easily to check if A is a consistent subset of S, then A^C is a subsemigroup of S. Opposite assertion, "If T is a subsemigroup of S, then T^C is a consistent subset of S." not holds in general. A consistent subsemigroup F of S will called a *filter* of S. In that case, the subset F^C is a completely prime ideal of S. The opposite assertion "If J is a completely prime ideal of S, then J^C is a filter of S." not holds in general.

In the Classical Semigroup Theory concept of positive quasi-order has been introduced by B.M.Schein. After that, positive quasi-orders have been studied from different points of view by many authors, mainly by T.Tamura [13-14], M.S.Putcha [6], and S.Bogdanovic and M.Ciric [4]. Quasi-antiorder relation in semigroups with apartness the first time defined by this author in his one earlier paper in 1996. Further investigation on basic properties of quasi-antiorder relations on sets and semigroups with apartness the author has did in his paper [10] and in forthcoming papers [11, 12]. Positive quasi-antiorders studied by Crvenkovic and Romano in their forthcoming paper [3].

For undefined notions and notations we referee to books [1-2, 5] and to papers [7 - 10].

In this paper we study a constructive aspect of construction of positive quasi-antiorders of semigroups with apartness.

2 Positive Quasi-antiorders

By a quasi-antiorder we mean a consistent cotransitive relation on a set. For a quasi-antiorder σ on a semigroup S we say that it is *compatible* with the semigroup operation if and only if

$$(\forall a, b, x \in S)((((ax, bx) \in \sigma \Longrightarrow (a, b) \in \sigma) \wedge ((xa, xb) \in \sigma \Longrightarrow (a, b) \in \sigma)).$$

A relation τ on a semigroup S is called *positive* if and only if

$$(\forall a, b \in S)((a, ab) \bowtie \tau \wedge (b, ab) \bowtie \tau),$$

and it is called *lower-potent* if
$(a^n, a) \bowtie \tau$, for any $a \in S$ and any $n \in \mathbf{N}$.

Clearly, if τ is a quasi-antiorder compatible with the semigroup operation, then it is lower-potent if and only if $(a^2, a) \bowtie \tau$, for all $a \in S$. Indeed: Suppose that $(a^2, a) \bowtie \tau$ for any $a \in S$ holds and suppose that $(a^k, a) \bowtie \tau$, for any $a \in S$ and $k \in \mathbf{N}$, and let (u, v) be an arbitrary element of τ. Then:
$(u, v) \in tau \Longrightarrow (u, a^{k+1}) \in \tau \vee (a^{k+1}, a^2) \in \tau \vee (a^2, a) \in \tau \vee (a, v) \in \tau$
$\Longrightarrow u \neq a^{k+1} \vee (a^k, a) \in \tau \vee (a^2, a) \in \tau a \neq v$
$\Longrightarrow (a^{k+1}, a) \neq (u, v) \in \tau$.

So, by induction, we have that the formula $(a^n, a) \bowtie \tau$ holds for any $a \in S$ and any $n \in \mathbf{N}$.

In the following we will describe a construction of maximal positive quasi-antiorder relation on a semigroup S. Let a and b be elements of S. Then, by Theorem 6 in [9], the set $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S such that:
(i) $a \bowtie C_{(a)}$;
(ii) $C_{(a)} \neq \emptyset \Longrightarrow 1 \in C_{(a)}$;
(iii) let a be an invertible element of S. Then $C_{(a)} = \emptyset$;
(iv) $(\forall x, y \in S)(C_{(a)} \subseteq C_{(xay)})$;
(v)$C_{(a)} \cup C_{(b)} \subseteq C_{ab}$.

The consistent subset $C_{(a)}$ is called a *principal* consistent subset of S generated by a. We introduce relation f on S, defined by $(a, b) \in f$ if and only if $b \in C_{(a)}$. The relation f has the following properties ([9], Theorem 7):
(a) f is a consistent relation on S;
(b) $(a, b) \in f \Longrightarrow (\forall x, y \in S)((xay, b) \in f)$;

(c) $(a,b) \in f \Longrightarrow (\forall n \in \mathbf{N})((a^n, b) \in f)$;
(d) $(\forall x, y \in S)((a, xby) \in f \Longrightarrow (a,b) \in f)$;
(e) $(\forall x, y \in S)\neg((a, xay) \in f)$.

For an element a of a semigroup S and for $n \in \mathbf{N}$ we introduce the following notations

$$A_n(a) = \{x \in S : (a,x) \in^n f\},$$

$$A(a) = \{x \in S : (a,x) \in c(f)\}$$

$$B_n(a) = \{y \in S : (y,a) \in^n f\},$$

$$B(a) = \{y \in S : (y,a) \in c(f)\}.$$

In the following two lemmas we will present some basic characteristic of these sets.

Lema 2.1 *Let a and b be elements of a semigroup S. Then*:

(1) *The set $A(a) = \cap_{n\in \boldsymbol{N}}$ is the maximal strongly extensional consistent subset of S such that $a \bowtie A(a)$.*

(2) $A(a) \cup A(b) \subseteq A(ab)$.

Proof immediately follows from Theorem 2 and Theorem 3 of the paper [10]. □

Symmetrically, we have:

Lemma 2.2 *Let a and b be elements of a semigroup S. Then*:

(1) *The set $B(a) = \cap_{n\in \boldsymbol{N}} B_n(a)$ is the maximal strongly extensional ideal of S such that $a \bowtie B(a)$*

(2) $B(ab) \subseteq B(a) \cap B(b)$.

Proof immediately follows from Theorem 4 and Theorem 5 of the paper [10]. □

In the next theorem we give a construction of the maximal positive quasi-antiorder relation in semigroup S.

Theorem 2.3 *The relation $c(f)$ is the maximal positive quasi-antiorder relation on semigroup S.*

Proof immediately follows from Lemmas 2.1 - 2.2 and Lemma 3.2 in [3]. □

Theorem 2.4 *A quasi-anti-order τ on a semigroup S is positive ifand only if it is contained in the maximal quasi-antiorder relation $c(f)$ on S.*

Proof: It is clear that if τ is a positive quasi-antiorder relation on S, then $\tau \subseteq c(f)$, since $c(f)$ is the maximal positive quasi-antiorder relation on S. Let $\tau \subseteq c(f)$. Then, $(x, xy) \bowtie c(f) \supseteq \tau$ and $(x, yx) \bowtie c(f) \supseteq \tau$ for any x, y of S. So, the quasi-antiorder τ is positive. □

Concluding note I: As mentioned in the Theorem 2.4, a quasi-antiorder on a semigroup S is positive if and only if it is contained in the maximal positive quasi-antiorder $c(f)$. So, the family of positive quasi-antiorders on S is a principal ideal of the family $\boldsymbol{Q}(S)$ of all quasi-antiorders on S gneerated by the relation $c(f)$ on S.

Besides, we will describe maximal lower-potent positive quasi-antiorder relation on a semigroup S. Let S be a semigroup with apartness and a, b arbitrary elements of S. As mentioned above in this section, the set $C_{(a)}$ is a consistent subset of S called a principal consistent subset of S generated by the element a. Set $cr(C_{(a)}) = \{x \in S : (\forall n \in \mathbf{N})(x^n \in C_{(a)})\}$ is called a *coradical* of principal consistent subset of S generated by the element a. We introduce a relation s on S, on the following way

$$(a, b) \in s \Longleftrightarrow b \in cr(C_{(a)})$$

and we will describe some properties of relations s and $c(s)$

Lemma 2.5 ([12], Lemma 2.1) *The relation $c(s)$ satisfies the following properties* :

(1) $c(s)$ is a consistent relation on S;
(2) $c(s)$ is a cotransitive relation ;
(3) $(\forall n \in \mathbf{N})((a, a^n) \bowtie c(s))$.

For an element a of a semigroup S and for $n \in \mathbf{N}$ we introduce the following notations:

$$A_n(a) = \{x \in S : (a, x) \in^n s\},$$
$$A(a) = \{x \in S : (a, x) \in c(s)\}$$
$$B_n(a) = \{x \in S : (x, a) \in^n s\},$$
$$B(a) = \{x \in S : (x, a) \in c(s)\}.$$

By the following results we will present some basic characteristics of these sets.

Lemma 2.6 ([12], Theorem 2.3) *Let a and b be elements of a semigroup S. Then*:

(1) *The set $A(a) = \cap_{n\in \boldsymbol{N}} A_n(a)$ is a maximal strongly extensional consistent potent semifilter of S such that $a \bowtie A(a)$.*
(2) $A(a) \cup A(b) \subseteq A(ab)$;
(3) $(\forall n \in \mathbf{N})(A(a) = A(a^n))$

Analogously, we have:

Lema 2.7 ([12], Theorem 2.3) *Let a and b be elements of a semigroup S. Then*:

(1) *The set $B(a) = \cap_{n\in \boldsymbol{N}} B_n(a)$ is the maximal strongly extensional completely potent semiprime ideal of S such that $a \bowtie B(a)$.*

(2) $B(ab) \subseteq B(a) \cap B(b)$;
(3) $(\forall n \in \mathbf{N})(B(a^n) = B(a))$.

Theorem 2.8 The relation $c(s)$ is the maximal positive quasi-antiorder relation on a semigroup S and the following $(\forall a \in S)(\forall n \in \mathbf{N})((a^n, a) \bowtie c(s))$ holds.
textbfProof immediately follows from Lemmas 2.6 - 2.7 and lemma 3.2 in [3]. □

Theorem 2.8 *A positive quasi-antiorder τ on a semigroup S is lower-potent positive if and only if it is contained in the maximal lower-potent positive quasi-antiorder on S.*

Proof: It is clear that if τ is a lover-potent positive quasi-antiorder relation on S, then $\tau \subseteq c(s)$, since $c(s)$ is the maximal lower-potent positive quasi-antiorder on S.
Let $\tau \subseteq c(s)$. Then $(x^n, x) \bowtie c(s) \supseteq \tau$ for any x of S and for any natural number n. So, the positive quasi-antiorder τ is lower-potent. □

Concluding note II:

(1) it is clear that the family $Q_{lp}(S)$ of lower-potent positive quasi-antiorders on S is completely lattice. Indeed: Let $\{\tau_k\}_{k\in J}$ beb a family of lower-potent positive quasi-antiorders on a semigroup S. Since $(\forall k \in J)(\forall n \in \mathbf{N})(\tau_k \bowtie (a^n, a))$, we have $\cup_{k\in J}\tau_k \bowtie (a^n, a)$. besides, $c(\cap_{k\in J}\tau_k)$ is a positive quasi-antiorder in S, and the following

$$c(\cap_{k\in J}\tau_k) \subseteq \cap_{k\in J}\tau_k \bowtie (a^n, a)$$

folds for any natural n. So, $Q_{lp}(S)$ is a completely lattice.

(2) As mentioned in the Theorem 2.9, a positive quasi-antiorder on a semigroup S is lower-potent positive quasi-antiorder relation on S if and only if it is contained in the maximal lower-potent positive quasi-antiorder $c(s)$. So, the family $Q_{lp}(S)$ of lower-potent positive quasi-antiorders on S is a principal ideal of $Q_p(S)$ generated by the relation $c(s)$ on S.

3 The Main Results

In the paper [14], Tamura turned a general method for construction of new relations from a given relation. Namely, to any relation ρ on a semigroup S he associated the relation $a(\rho)$ on S defined by $(a, b) \in a(\rho) \iff (\exists n \in \mathbf{N})((a, b^n) \in \rho)$, and the transitive closure of $a(\rho)$. Expect in the Tamura's paper, such constructed relations have been intensively used in a series of papers by M.S.Putcha.

Using this ideal, in this paper for a given relation σ in a semigroup S with apartness we construct the following relation:

$$(a,b) \in p(\sigma) \Longleftrightarrow (\forall n \in \mathbf{N})((a,b^n) \in \sigma).$$

A sense of such method is to build a lower-potent quasi-antiorder from a given relation σ. namely, the following assertion can be easily proved:

Theorem 3.1 *The maximal lower-potent quasi-antiorder on a semigroup S contained in σ on S equals $cp(\sigma \cap \neq)$.*

Proof: It is clear that $cp(\sigma \cap \neq)$ is a consistent and cotrasitive in semigroup S. By [7], $cp(\sigma \cap \neq)$ is the maximal quasi-antiorder relation under $\sigma \cap \neq$. let u, v and a be arbitrary elements of S such that $(u,v) \in cp(\sigma \cap \neq)$. Then:

$$(u,v) \in cp(\sigma \cap \neq) \Longrightarrow (u,a^n) \in cp(\sigma \cap \neq) \vee (a^n,a) \in cp(\sigma \cap \neq) \vee (a,v) \in cp(\sigma \cap \neq)$$

for any natural n. Thus, for any natural n, we have:
$(u,v) \in cp(\sigma \cap \neq) \Longrightarrow u \neq a^n \vee (a^n,a) \in cp(\sigma \cap \neq) \vee a \neq$
$\Longrightarrow (u,v) \neq (a^n,a) \vee (\forall m \in \mathbf{N})((a^n,a^m) \in (\sigma \cap \neq))$
$\Longrightarrow (a^n,a) \neq (u,v) \in cp(\sigma \cap \neq)$.

Let τ be a positive quasi-antiorder under σ. Then $\tau \subseteq \neq$ and $\tau \subseteq c(\sigma)$. Thus, $\tau \subseteq \sigma \cap \neq$. Suppose that $\tau \subseteq^n (\sigma \cap \neq)$ for some $n \in \mathbf{N}$. Hence,

$$\tau \subseteq \tau * \tau \subseteq (\sigma \cap \neq) *^n (\sigma \cap \neq) =^{n+1} (\sigma \cap \neq).$$

So, by induction, we have $\tau \subseteq \cap^n_{n \in \mathbf{N}} (\sigma \cap \neq)$. Therefore, the relation $cp(\sigma \cap \neq)$ is the maximal lower-potent quasi-antiorder under $\sigma \cap \neq$. □

On the other hand, a good characteristic of this construction is that it supplies positiveness property, so the following theorem can be easily verified:

Theorem 3.2 The maximal lower-potent positive quasi-antiorder on a semigroup S contained in a relation σ on S equals $cp(\sigma \cap f)$

Proof: Let (u,v) ne an arbitrary element of $cp(\sigma \cap f)$ and let a, b be elements of S. Then:
$(u,v) \in cp(\sigma \cap f) \Longrightarrow (u,a) \in cp(\sigma \cap f) \vee (a,ab) \in cp(\sigma \cap f) \vee (ab,v) \in cp(\sigma \cap f)$
$\Longrightarrow u \neq a \vee (a,ab) \in cp(\sigma \cap f) \vee ab \neq v$
$\Longrightarrow (a,ab) \neq (u,v) \in cp(\sigma \cap f)$.

For $(b,ab) \bowtie cp(\sigma \cap f)$ the proof is analogous. So, $cp(\sigma \cap f)$ is a positive quasi-antiorder under σ.

Let τ be a positive quasi-antiorder under σ. Then, $\tau \subseteq c(f)$ and $\tau \subseteq c(\sigma)$. Thus, $\tau \subseteq \sigma \cap f$. Suppose that $\tau \subseteq^n (\sigma \cap f)$ for some $n \in \mathbf{N}$. Hence,

$$\tau \subseteq \tau * \tau \subseteq (\sigma \cap f) *^n (\sigma \cap f) =^{n+1} (\sigma \cap f).$$

So, by induction, we have $\tau \subseteq \cap_{n\in\mathbf{N}}^{n}(\sigma\cap f)$. Therefore, the relation $cp(\sigma\cap f)$ is the maximal positive quasi-antiorder under σ. $\square$

Therefore, if we start from a consistent and positive relation τ, then withouth serious problems we obtain that $cp(\tau)$ is a positive lower-potent quasi-antiorder. But, if we want to obtain that $cp(\tau)$ is compatible with the semigroup operation, then more serious difficulties are arising. Namely, there is a following construction

$$cp(\tau)^{+} = \{(a, b) \in S \times S : (\exists u, v \in S^{1})((uav, ubv) \in cp(\tau))\}$$

-a minimal extension of $cp(\tau)$ - such that it is compatible with the semigroup operation, but with losing of substantial characteristics. So, the problem of compatibility of relation $cp(\tau)$ with the semigroup operation is open.

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